Sharp Thresholds and Characteristic Shells for a Two-Sided Clifford Pencil

Aaron Nagy
Independent researcher, South Bend, Indiana, USA
aaronnagy1987@gmail.com

Preprint, July 2026

Abstract

Let $V = \mathbb{R}e_0 \oplus V_\perp$, $\dim V_\perp = d$, and let $\mathrm{Cl}_{d+1,\mathbb{C}}$ be the complex Euclidean Clifford algebra. For

$$m_y(A,B) = \tfrac{1}{2}[A,B] + iy(AB+BA), \qquad y \in \mathbb{R}\setminus\{0\},$$

we study the finite-dimensional pencil

$$P_{a,y,\eta}(u) = uD_y + aI + \tau - \eta u G_y \Gamma_y$$

on $V_\perp^* \otimes \mathrm{Cl}_{d+1,\mathbb{C}}$, where τ is Clifford reversion, $|a|<1$, and u is a complex parameter. We give the complete 2×2 Schur reduction indexed by transverse Clifford grade. The exact threshold is the scalar coefficient

$$C(d,\eta,Y) = (1+\eta d)(1-4\eta d Y), \qquad Y = y^2.$$

If $C>0$, the pencil is invertible for every real u and, more strongly, for every u with $\mathrm{Re}(u^2)\geq 0$, with $\|P(u)^{-1}\| \leq K/(1+|u|)$. If $C=0$, the inverse norm is comparable to $|u|$ along the real axis. We also compute the exact degree lost by the full determinant at infinity. The two components of the critical locus have different principal nullities, which we compute exactly. If $C<0$, the real characteristic set is exactly $\{-u_*, u_*\}$, where

$$u_*^2 = \frac{1-a^2}{-C(d,\eta,Y)}.$$

The characteristic kernel has dimension one for odd d and two for even d; we give explicit generators, an exact full-determinant factorization, and show that the determinant crosses the shell with order one and two, respectively. In both parities the inverse nevertheless has only a simple pole; its residue has image equal to the characteristic kernel. For even transverse grades, positivity follows from an exact stabilizer identity. For odd grades, the middle coefficient of the Schur numerator has a Bernstein-positive decomposition. Thus the scalar obstruction is the sharp threshold for the entire normalized scalar/reversion family.

Keywords: Clifford algebra, matrix pencil, parameter ellipticity, Schur complement, characteristic shell, resolvent estimate
2020 Mathematics Subject Classification: Primary 15A22, 15A66; Secondary 35J46, 47A56

1 Introduction

Clifford-valued first-order symbols often decompose into generalized-gradient or representation-theoretic blocks; see, for example, [2, 3, 4]. The pencil considered here is a specific finite-dimensional frozen symbol built from a two-sided Clifford product, reversion, and a rank-type return term. The purpose of the paper is not to classify general Clifford pencils. It is to determine, exactly and in all dimensions, when this explicit family has a real characteristic shell and when it satisfies a parameter-elliptic estimate in the classical sense of elliptic problems with a parameter [1].

A scalar Schur block immediately exposes the coefficient $(1+\eta d)(1-4\eta d y^2)$. The nontrivial question is whether another Clifford grade can become singular before the scalar block does. We answer no. This converts a candidate scalar obstruction into the exact threshold for the full exponentially large Clifford system. The parity structure is delicate: even grades are stabilized by a manifestly nonnegative correction, while odd grades require a positive interpolation identity in the zero-order parameter a.

The shell calculation also separates two notions that are easily conflated. For odd transverse dimension the determinant has a simple shell zero and the kernel is one-dimensional. For even transverse dimension two distinct grade blocks vanish, so the determinant has order two and the kernel is two-dimensional; nevertheless the inverse still has only a first-order pole, with rank-two residue. Thus determinant multiplicity records the number of crossing blocks, while the resolvent order records their transversal first-order crossing. This distinction is one of the main structural outputs of the model.

The proof has four layers. First, the principal return operator is diagonalized on the Clifford monomial basis, which identifies the two components of the critical locus and their different multiplicities. Second, a rectangular matrix-determinant reduction transfers the full pencil to two-dimensional Schur blocks indexed by transverse grade. Third, exact positivity identities show that the scalar block is the first and only source of finite real characteristics. Finally, the same blocks determine the critical growth at infinity and the Laurent residue at the characteristic shell. This organization also separates the finite-dimensional Clifford calculation from the terminology of parameter ellipticity used to describe its resolvent consequence.

2 Clifford conventions and the pencil

Let $V = \mathbb{R}e_0 \oplus V_\perp$ be Euclidean with orthonormal basis $e_0, e_1, \dots, e_d$. Let $\gamma_0, \dots, \gamma_d$ generate $\mathrm{Cl}_{d+1,\mathbb{C}}$ with

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

For a homogeneous blade X of grade r, Clifford reversion is

$$\tau X = \rho_r X, \qquad \rho_r = (-1)^{r(r-1)/2}.$$

For real $y \neq 0$, define

$$m_y(A,B) = \frac{1}{2}[A,B] + iy(AB+BA).$$

If a blade A commutes with a monomial X, then $m_y(A,X) = 2iyAX$; if it anticommutes, then $m_y(A,X) = AX$.

Set

$$D_y = m_y(\gamma_0,\cdot), \qquad G_{i,y} = m_y(\gamma_i,\cdot), \qquad \Gamma_{i,y} = m_y(\gamma_0\gamma_i,\cdot).$$

Let

$$E = V_\perp^* \otimes \mathrm{Cl}_{d+1,\mathbb{C}}\,.$$

We equip the Clifford monomials and the tensor-product basis of E with the standard Hermitian inner product; all operator norms and singular values below refer to this choice. Since the spaces are finite-dimensional, the qualitative estimates are unchanged under any equivalent norm. Let D_y act componentwise on E, and define

$$(G_y s)_i = G_{i,y} s, \qquad \Gamma_y(v) = \sum_{i=1}^{d} \Gamma_{i,y} v_i.$$

For $a, \eta \in \mathbb{R}$, $|a| < 1$, consider

$$P_{a,y,\eta}(u) = uD_y + aI + \tau - \eta u G_y \Gamma_y.$$

We write

$$Y = y^2 > 0, \qquad q = 1 - a^2 > 0, \qquad w = u^2.$$

3 The principal return spectrum

The high-frequency operator on E is

$$A_{y,\eta} = D_y - \eta G_y \Gamma_y,$$

where D_y acts componentwise. The associated return operator on the Clifford algebra is

$$T_y = \Gamma_y D_y^{-1} G_y.$$

Since D_y is invertible, the Sylvester determinant identity gives

$$\det A_{y,\eta} = \det(D_y|_E)\det(I - \eta T_y), \tag{1}$$

and $A_{y,\eta}$ is invertible if and only if $I - \eta T_y$ is invertible.

Proposition 3.1 (principal return spectrum)**.** *For $Y > 0$, the operator T_y is diagonal on the monomial basis*

$$X_{\epsilon,I} = \gamma_0^\epsilon \gamma_I, \qquad \epsilon \in \{0,1\}, \quad I \subseteq \{1,\dots,d\}.$$

Writing $k = |I|$, its eigenvalue is

$$\lambda_{\epsilon,k}(Y) = \begin{cases} -d, & k \text{ odd}, \\ 4Y(d-k) - k, & k \text{ even}, \ \epsilon = 0, \\ 4Yk - (d-k), & k \text{ even}, \ \epsilon = 1. \end{cases}$$

Proof. For fixed i, the composition $\Gamma_{i,y} D_y^{-1} G_{i,y}$ returns every monomial to itself because the left Clifford factor is $(\gamma_0\gamma_i)\gamma_0\gamma_i = -1$. Its scalar coefficient is the product of three factors, each equal to 1 or $2iy$ according to a commutation parity. If k is odd, every transverse index contributes -1. If k is even and $\epsilon = 0$, indices outside I contribute $4Y$ and indices inside I contribute -1. For $\epsilon = 1$, the two contributions are interchanged. Summing over i gives the formula. □

Corollary 3.2 (principal determinant and critical multiplicities)**.** *One has*

$$\frac{\det A_{y,\eta}}{\det(D_y|_E)} = \prod_{\substack{0\le k\le d\\ k \text{ odd}}} (1+\eta d)^{2\binom{d}{k}} \times \prod_{\substack{0\le k\le d\\ k \text{ even}}} \big(1-\eta(4Y(d-k)-k)\big)^{\binom{d}{k}} \big(1-\eta(4Yk-(d-k))\big)^{\binom{d}{k}}.$$

In particular, if $C(d,\eta,Y)>0$, then $A_{y,\eta}$ is invertible. On the critical locus $C=0$, its nullity is

$$\dim\ker A_{y,\eta} = \begin{cases} 1+\mathbf{1}_{2|d}, & \eta=(4dY)^{-1},\\ 2^d+1+\mathbf{1}_{2|d}, & \eta=-d^{-1}. \end{cases}$$

Proof. Insert the eigenvalues of Proposition 3.1 into (1), with multiplicity $\binom{d}{k}$ for each value of ϵ. For even k, both eigenvalues lie in the interval $[-d, 4Yd]$. If $C>0$, then $1/\eta$ lies outside this interval, with the usual interpretation when $\eta=0$, so no factor vanishes.

If $\eta=(4dY)^{-1}$, the eigenvalue $4Yd$ occurs for $(\epsilon,k)=(0,0)$ and also for $(1,d)$ when d is even. If $\eta=-d^{-1}$, the eigenvalue $-d$ occurs for both values of ϵ at every odd k, for $(\epsilon,k)=(1,0)$, and additionally for $(0,d)$ when d is even. The binomial identity $\sum_{k\text{ odd}}\binom{d}{k}=2^{d-1}$ gives the stated nullities. □

4 Exact Schur blocks

Put

$$B(u) = uD_y + aI + \tau.$$

For a transverse blade γ_J of grade r, the space

$$E_J = \operatorname{span}_{\mathbb{C}}\{\gamma_J, \gamma_0\gamma_J\}$$

is $B(u)$-invariant, and in the displayed basis

$$B_r(u) = \begin{pmatrix} a+\rho_r & c_r u \\ c_r u & a+\rho_{r+1}\end{pmatrix}, \qquad c_r = \begin{cases} 2iy, & r \text{ even},\\ 1, & r \text{ odd}. \end{cases}$$

Define

$$\delta_\pm(w) = (a\pm 1)^2 + 4Yw, \qquad \Delta(w) = q + w.$$

Then

$$\det B_r(u) = \begin{cases} \delta_+(w), & r\equiv 0 \pmod 4,\\ \delta_-(w), & r\equiv 2 \pmod 4,\\ -\Delta(w), & r \text{ odd}. \end{cases}$$

In particular, $B(u)$ is invertible whenever $\operatorname{Re} w \ge 0$.

The matrix determinant lemma gives

$$P = B - \eta u G_y \Gamma_y$$

and reduces invertibility to the Schur complement

$$S(u) = I - \eta u \Gamma_y B(u)^{-1} G_y$$

on $\mathrm{Cl}_{d+1,\mathbb{C}}$.

Lemma 4.1 (rectangular determinant and kernel reduction)**.** *Whenever $B(u)$ is invertible,*

$$\det P(u) = \det B(u) \det S(u).$$

If in addition $\eta u \neq 0$, then G_y is injective and

$$z \longmapsto B(u)^{-1} G_y z$$

identifies $\ker S(u)$ *with* $\ker P(u)$ *up to the harmless nonzero scalar ηu in the inverse correspondence.*

Proof. Apply Sylvester's identity $\det(I - UV) = \det(I - VU)$ to $U = \eta u B(u)^{-1} G_y$ and $V = \Gamma_y$. Each $G_{i,y}$ maps every Clifford monomial to a nonzero scalar multiple of another monomial because $y \neq 0$; hence the stacked map G_y is injective. If $S(u)z = 0$, then

$$P(u)B(u)^{-1}G_y z = G_y S(u) z = 0.$$

Conversely, if $P(u)x = 0$, then $x = \eta u B(u)^{-1} G_y \Gamma_y x$ and $z = \eta u \Gamma_y x$ lies in $\ker S(u)$. These maps are inverse after the stated scalar normalization. □

Lemma 4.2 (grade-two reduction)**.** *For every transverse blade γ_I, $|I| = k$, the Schur complement preserves*

$$E_I = \operatorname{span}_{\mathbb{C}}\{\gamma_I, \gamma_0 \gamma_I\}.$$

The resulting matrix depends only on k and k mod 4*.*

Proof. Fix i. Left multiplication by γ_i toggles the ith transverse index, so $G_{i,y}$ sends each basis vector of E_I into the two-dimensional space attached to $I \triangle \{i\}$. The operator $B(u)$ preserves that space, because both D_y and reversion interchange or rescale its two basis blades. After applying $B(u)^{-1}$, left multiplication by $\gamma_0\gamma_i$ toggles the same transverse index back, so $\Gamma_{i,y}$ returns the vector to E_I. There are only two contribution types, according as $i \in I$ or $i \notin I$, with multiplicities k and $d - k$. Their signs are determined by the reversion signs in grades $k-1, k, k+1$, and therefore by k mod 4. Summing the contributions proves the invariant-block statement and the asserted dependence. □

4.1 Even grades

Suppose k is even and put

$$\varepsilon_k = (-1)^{k/2},$$

$$\lambda_{0,k} = 4Y(d-k) - k, \qquad \lambda_{1,k} = 4Yk - (d-k).$$

A direct multiplication using the matrices B_{k-1}^{-1} and B_{k+1}^{-1} gives

$$\Gamma_y B^{-1} G_y\big|_{E_I} = \frac{1}{\Delta(w)} \begin{pmatrix} u\lambda_{0,k} & 2iy\big(\varepsilon_k(2k-d) - ad\big) \\ 2iy\big(-\varepsilon_k(2k-d) - ad\big) & u\lambda_{1,k} \end{pmatrix}. \tag{2}$$

Proposition 4.3 (even-grade stabilizer identity)**.** *Let*

$$C = (1 + \eta d)(1 - 4\eta d Y).$$

For every even k, after clearing the positive denominator $\Delta(w)^2$, the Schur determinant is

$$\det S_k(u) = \frac{N_k^{\mathrm{ev}}(w)}{\Delta(w)^2},$$

where

$$N_k^{\mathrm{ev}}(w) = (q + Cw)(q + w) + \eta^2 k(d-k)\big((1+4Y)^2 w^2 + 16Yw\big). \tag{3}$$

For $k = 0$, one factor cancels and

$$\det S_0(u) = \frac{q + Cw}{q + w}. \tag{4}$$

If d is even, the same cancellation occurs in the terminal block $k = d$, so $\det S_d(u) = \det S_0(u)$.

Proof. Insert (2) into $S_k = I - \eta u \Gamma B^{-1} G$. Expanding the determinant gives

$$(q + w - \eta w\lambda_{0,k})(q + w - \eta w\lambda_{1,k}) + 4\eta^2 Yw\big(\varepsilon_k(2k-d) - ad\big)\big(-\varepsilon_k(2k-d) - ad\big).$$

Using $\varepsilon_k^2 = 1$ and collecting coefficients yields (3). At $k = 0$, the correction vanishes and the factor $q + w$ cancels. When d is even and $k = d$, the correction again vanishes and the two diagonal factors are interchanged, giving the same quotient. □

4.2 Odd grades

Now suppose $0 < k < d$ is odd and put

$$\sigma_k = (-1)^{(k-1)/2} \in \{1, -1\}.$$

For $\sigma \in \{1, -1\}$, write $\delta_\sigma = \delta_+$ when $\sigma = 1$ and $\delta_\sigma = \delta_-$ when $\sigma = -1$. Define

$$A_k(w) = \frac{k}{\delta_{\sigma_k}(w)} + \frac{d-k}{\delta_{-\sigma_k}(w)},$$

$$b_k(w) = \frac{k(a + \sigma_k)}{\delta_{\sigma_k}(w)} - \frac{4Y(d-k)(a - \sigma_k)}{\delta_{-\sigma_k}(w)},$$

$$c_k(w) = -\frac{4Yk(a + \sigma_k)}{\delta_{\sigma_k}(w)} + \frac{(d-k)(a - \sigma_k)}{\delta_{-\sigma_k}(w)}.$$

The exact return block is

$$\Gamma_y B^{-1} G_y\big|_{E_I} = \begin{pmatrix} -4uYA_k(w) & b_k(w) \\ c_k(w) & -4uYA_k(w) \end{pmatrix}. \tag{5}$$

Proposition 4.4 (odd-grade Bernstein positivity)**.** *For $0 < k < d$ odd,*

$$\det S_k(u) = \frac{N_k^{\mathrm{odd}}(w)}{\delta_+(w)\delta_-(w)},$$

where

$$N_k^{\mathrm{odd}}(w) = 16Y^2(1 + \eta d)^2 w^2 + B_k(a, Y)w + q^2.$$

The middle coefficient has the exact decomposition

$$\begin{aligned} B_k(a, Y) = \frac{(1-a)^2}{4} B_k(-1, Y) + \frac{(1+a)^2}{4} B_k(1, Y) \\ + (1 - a^2)\eta^2 k(d-k)(16Y^2 + 1), \end{aligned} \tag{6}$$

with

$$\{B_k(-1, Y), B_k(1, Y)\} = \{16Y(1 + \eta k)^2, 16Y(1 + \eta(d-k))^2\}.$$

Hence

$$B_k(a, Y) > 0$$

for every $|a| < 1$, $Y > 0$, and $\eta \in \mathbb{R}$.

Proof. Substitute (5) into S_k and clear $\delta_+\delta_-$. The leading and constant coefficients are immediate. The coefficient of w is a quadratic polynomial in a. Evaluating it at $a = \pm 1$ gives the two displayed squares, and subtracting their quadratic Bernstein interpolation leaves exactly $(1-a^2)\eta^2 k(d-k)(16Y^2+1)$. All three terms in (6) are nonnegative. If $\eta \neq 0$, the final term is strictly positive; if $\eta = 0$, both endpoint squares are positive. This proves the strict positivity statement. □

Proposition 4.5 (top odd grade)**.** *If d is odd and $k = d$, then*

$$\det S_d(u) = \frac{(a+\sigma_d)^2 + 4Y(1+\eta d)^2 w}{(a+\sigma_d)^2 + 4Yw}.$$

This block has no zero with $\operatorname{Re} w \geq 0$.

Proof. For $k = d$, only the first denominator in (5) occurs. Substitution into $S_d = I - \eta u \Gamma B^{-1} G$ and a direct 2×2 determinant expansion give the displayed quotient. Its numerator and denominator are affine polynomials in w with nonnegative real coefficients and strictly positive constant term because $|a| < 1$. □

Proposition 4.6 (full determinant factorization)**.** *For $0 \leq r \leq d$, put*

$$b_r(w) = \begin{cases} \delta_+(w), & r \equiv 0 \pmod 4, \\ \delta_-(w), & r \equiv 2 \pmod 4, \\ -\Delta(w), & r \text{ odd}, \end{cases} \qquad \mathcal{B}_d(w) = \prod_{r=0}^{d} b_r(w)^{\binom{d}{r}}.$$

Let $s_k(w) = \det S_k(u)$, using Propositions 4.3–4.5. Then, as an identity of rational functions and hence by continuation as a polynomial identity,

$$\det P_{a,y,\eta}(u) = \mathcal{B}_d(u^2)^d \prod_{k=0}^{d} s_k(u^2)^{\binom{d}{k}}. \tag{7}$$

In particular, every finite characteristic multiplicity is obtained by summing the vanishing orders of the explicit grade blocks with their binomial multiplicities.

Proof. The operator $B(u)$ acts componentwise on the d copies of the Clifford algebra in E. On the Clifford algebra it is the direct sum of one block $B_r(u)$ for each transverse blade of grade r, so

$$\det(B(u)|_E) = \mathcal{B}_d(w)^d.$$

The Schur complement is the direct sum of the two-dimensional blocks $S_k(u)$, one for each transverse blade of grade k, and therefore

$$\det S(u) = \prod_{k=0}^{d} s_k(w)^{\binom{d}{k}}.$$

The matrix determinant lemma gives (7) wherever $B(u)$ is invertible. Both sides define the same polynomial after cancellation, so the identity holds for every u. □

Corollary 4.7 (degree loss at the two critical branches)**.** *Let*

$$N = d\,2^{d+1} = \dim_{\mathbb{C}} E.$$

Away from the critical locus, $\det P_{a,y,\eta}(u)$ *has degree* N *as a polynomial in* u*. On the branch* $\eta = (4dY)^{-1}$ *its degree is*

$$N - 2\big(1 + \mathbf{1}_{2|d}\big),$$

and on the branch $\eta = -d^{-1}$ *its degree is*

$$N - \big(2^d + 2 + 2\mathbf{1}_{2|d}\big).$$

Thus the two components of $C = 0$ *are distinguished not only by their principal nullities but also by the exact order lost at infinity.*

Proof. The background determinant in (7) has degree N in u. At $\eta = (4dY)^{-1}$, exactly the scalar block $k = 0$ and, for even d, the terminal block $k = d$ decay as $w^{-1} = u^{-2}$; every other Schur determinant has a nonzero limit. This gives the first loss.

At $\eta = -d^{-1}$, the scalar block, every odd-grade block, and the terminal even block when d is even decay as w^{-1}. Their multiplicity count is

$$1 + \sum_{k \text{ odd}} \binom{d}{k} + \mathbf{1}_{2|d} = 1 + 2^{d-1} + \mathbf{1}_{2|d}.$$

Each contributes two lost powers of u, proving the second formula. The explicit block numerators show that all stated leading coefficients are nonzero. □

5 The sharp threshold

Theorem 5.1 (sharp real-axis trichotomy)**.** *Let* $d \geq 1$*,* $Y > 0$*, and* $|a| < 1$*, and define*

$$C(d, \eta, Y) = (1 + \eta d)(1 - 4\eta dY).$$

Then:

1. *If* $C > 0$*,* $P_{a,y,\eta}(u)$ *is invertible for all real* u *and there is* $K > 0$ *such that*

$$\|P_{a,y,\eta}(u)^{-1}\| \leq \frac{K}{1 + |u|}.$$

2. *If* $C = 0$*, the pencil remains nonsingular for each finite real* u*, and there are constants* $c, K, U > 0$ *such that*

$$c|u| \leq \|P_{a,y,\eta}(u)^{-1}\| \leq K|u| \qquad (|u| \geq U).$$

   Thus the critical loss is exactly linear.

3. *If* $C < 0$*, the real characteristic set is exactly*

$$\{-u_*, u_*\}, \qquad u_*^2 = \frac{1 - a^2}{-C(d, \eta, Y)}.$$

   At each characteristic point,

$$\dim_{\mathbb{C}} \ker P_{a,y,\eta}(u) = 1 + \mathbf{1}_{2|d},$$

   where $\mathbf{1}_{2|d}$ *is* 1 *for even* d *and* 0 *for odd* d*.*

Proof. For $C < 0$, (4) vanishes at the displayed positive value of w, while $B(u_*)$ is invertible. The even-grade correction term in (3) is strictly positive for $0 < k < d$, and every odd-grade numerator is strictly positive for $w > 0$. Thus the scalar block $k = 0$ is always singular, and when d is even the terminal even block $k = d$ has the same vanishing determinant; no other block is singular. The relevant off-diagonal entries are nonzero because $u_* y\eta \neq 0$ and $|a| < 1$, so each singular 2×2 Schur block has rank one. The Schur-complement kernel correspondence gives the stated parity-dependent kernel dimension and shows that the real characteristic set is exactly $u = \pm u_*$.

Assume $C > 0$. For every even k, (3) is strictly positive for $w \geq 0$. For every interior odd k, the leading and constant coefficients of N_k^{odd} are nonnegative and positive respectively, while Proposition 4.4 gives a strictly positive middle coefficient. The top odd block is covered by Proposition 4.5. Thus every Schur block and hence $P(u)$ is invertible for finite real u.

The condition $C > 0$ also makes the principal operator $A_{y,\eta}$ invertible. Indeed, by Proposition 3.1, for $\eta > 0$ one has $4Yd < 1/\eta$, so every even eigenvalue is $< 1/\eta$, while for $-1/d < \eta \leq 0$ every principal eigenvalue is at least $-d$. The odd factor is $1 + \eta d > 0$ in both cases. Therefore

$$\frac{P(u)}{u} \longrightarrow A_{y,\eta} \qquad (u \to \pm\infty).$$

After compactifying the real axis, the least singular value of $P(u)/(1 + |u|)$ is a positive continuous function. This proves the estimate.

Finally suppose $C = 0$. Equation (4) gives $\det S_0 = q/(q + w)$. The explicit even block (2) shows that $S_0(u)$ converges to a nonzero rank-one matrix. Hence its largest singular value stays bounded above and below, while its least singular value is comparable to w^{-1}. Choose unit vectors $z_u \in E_\varnothing$ with $\|S_0(u)z_u\| = O(w^{-1})$ and set

$$x_u = B(u)^{-1} G_y z_u.$$

Since D_y and each $G_{i,y}$ are invertible on the Clifford monomial basis, $\|x_u\| \geq c_0/|u|$ for large $|u|$. Moreover,

$$P(u)x_u = G_y\big(I - \eta u \Gamma_y B(u)^{-1} G_y\big) z_u = G_y S_0(u) z_u,$$

so $\|P(u)x_u\| = O(w^{-1})$. Normalizing x_u gives $s_{\min}(P(u)) = O(|u|^{-1})$ and therefore the lower bound $\|P(u)^{-1}\| \geq c|u|$.

For the reverse inequality, the explicit determinant formulas show that every critical Schur block has determinant bounded below by a positive multiple of w^{-1} for large w, while its entries remain bounded; every noncritical block has a uniformly invertible limit. Consequently $\|S(u)^{-1}\| = O(w)$. The Woodbury formula

$$P^{-1} = B^{-1} + \eta u B^{-1} G_y S^{-1} \Gamma_y B^{-1}$$

and $\|B(u)^{-1}\| = O(|u|^{-1})$ give $\|P(u)^{-1}\| = O(|u|)$. Finite real u remain nonsingular because all Schur numerators have positive constant terms and, at $C = 0$, the remaining coefficients in Propositions 4.3–4.5 are positive. □

Proposition 5.2 (explicit shell kernels and crossing order)**.** *Assume $C(d, \eta, Y) < 0$, let $u \in \{-u_*, u_*\}$, and put $w_* = u^2$. Define*

$$\alpha_0 = q + (1 - 4\eta d Y) w_*, \qquad \beta_0 = 2i\eta d y (1 + a) u,$$

$$z_0 = \beta_0\, 1 - \alpha_0 \gamma_0, \qquad x_0 = B(u)^{-1} G_y z_0.$$

Then $x_0 \neq 0$ and spans the scalar-grade contribution to $\ker P(u)$*. If d is odd,* $\ker P(u) = \mathrm{span}_{\mathbb{C}}\{x_0\}$.

If d is even, set

$$\omega = \gamma_1 \cdots \gamma_d, \qquad \varepsilon = (-1)^{d/2},$$
$$\alpha_d = q + (1 + \eta d)w_*, \qquad \beta_d = -2i\eta dy(\varepsilon - a)u,$$
$$z_d = \beta_d \omega - \alpha_d \gamma_0 \omega, \qquad x_d = B(u)^{-1} G_y z_d.$$

Then

$$\ker P(u) = \operatorname{span}_{\mathbb{C}}\{x_0, x_d\}.$$

Moreover the zero of $\det P(u)$ *at each shell point has order one when* d *is odd and order two when* d *is even.*

Proof. Multiplying the scalar Schur block by $q + w$ gives, in the basis $(1, \gamma_0)$,

$$\begin{pmatrix} q + (1 - 4\eta dY)w & 2i\eta dy(1 + a)u \\ -2i\eta dy(1 - a)u & q + (1 + \eta d)w \end{pmatrix}.$$

At $w = w_*$ its determinant is zero, and the column vector $(\beta_0, -\alpha_0)^T$ lies in its kernel. Since $C < 0$ forces $\eta uy \neq 0$, this vector is nonzero. The Schur-complement correspondence $x = B^{-1}G_y z$ therefore gives the nonzero vector $x_0 \in \ker P(u)$.

When d is even, the terminal block $k = d$ in the basis $(\omega, \gamma_0\omega)$ becomes

$$\begin{pmatrix} q + (1 + \eta d)w & -2i\eta dy(\varepsilon - a)u \\ 2i\eta dy(\varepsilon + a)u & q + (1 - 4\eta dY)w \end{pmatrix}.$$

The vector $(\beta_d, -\alpha_d)^T$ is therefore its kernel vector. The two vectors belong to distinct transverse-grade summands. All other Schur blocks are invertible by the proof of Theorem 5.1, so the displayed vectors span the full kernel.

Finally,

$$\det S_0(u) = \frac{q + Cu^2}{q + u^2},$$

and the terminal even block has the same determinant. At $u = \pm u_*$,

$$\frac{d}{du} \det S_0(u) = \frac{2Cu}{q + u^2} \neq 0.$$

The background block $B(u)$ and every other Schur block remain invertible at the shell. The determinant lemma therefore gives one simple factor in odd d and two independent simple factors in even d. □

Corollary 5.3 (simple resolvent pole on the shell)**.** *Under the hypotheses of Theorem 5.2, let* $u_0 \in \{-u_*, u_*\}$*. There are a holomorphic matrix function* $H_{u_0}(u)$ *and a nonzero operator* R_{u_0} *such that*

$$P_{a,y,\eta}(u)^{-1} = \frac{R_{u_0}}{u - u_0} + H_{u_0}(u)$$

for u *near* u_0*,* $u \neq u_0$*. Moreover*

$$\operatorname{rank} R_{u_0} = 1 + \mathbf{1}_{2|d}, \qquad \operatorname{im} R_{u_0} = \ker P_{a,y,\eta}(u_0).$$

Thus even when d *is even and* $\det P$ *has a double zero, the resolvent pole is simple, with rank-two residue.*

Proof. The pencil is analytic in u. At u_0, Theorem 5.2 gives nullity $k = 1 + \mathbf{1}_{2|d}$ and shows that the determinant has a zero of exactly the same order k. Choose constant domain and range splittings so that the lower-right block of $P(u_0)$ is invertible and the upper-left block is $k \times k$. Analytic block elimination reduces $P(u)$ to the direct sum of an invertible analytic block and a Schur complement $F(u)$ with $F(u_0) = 0$. Writing $t = u - u_0$, one has

$$F(u) = tF_1 + O(t^2).$$

Every entry of F is divisible by t, while the determinant of F has order exactly k; hence $\det F_1 \neq 0$. Therefore $F(u)^{-1} = t^{-1}F_1^{-1} + O(1)$, and undoing the analytic block elimination shows that $P(u)^{-1}$ has a simple pole with residue of rank k. The coefficient of t^{-1} in $P(u)P(u)^{-1} = I$ is annihilated by $P(u_0)$, so its image lies in the kernel. Equality follows because both spaces have dimension k. □

Theorem 5.4 (sectorial parameter ellipticity)**.** *If $C(d, \eta, Y) > 0$, then $P_{a,y,\eta}(u)$ is invertible for every complex u satisfying*

$$\operatorname{Re}(u^2) \geq 0.$$

On this closed conic set,

$$\|P_{a,y,\eta}(u)^{-1}\| \leq \frac{K}{1 + |u|}$$

for some $K > 0$.

Proof. Let $w = u^2$. For even grades, (3) is a quadratic or linear polynomial in w with positive coefficients. For odd grades, the same is true by Proposition 4.4; the top odd block is explicit. A linear polynomial with positive coefficients has its zero on the negative real axis. A quadratic $Aw^2 + Bw + Q$ with $A, B, Q > 0$ has both real zeros negative or a complex-conjugate pair with real part $-B/(2A) < 0$. Hence no Schur numerator vanishes when $\operatorname{Re} w \geq 0$. The denominators Δ and $\delta_\pm$ also have zeros only on the negative real w-axis. This proves finite-parameter invertibility.

For large $|u|$, $P(u)/u$ converges uniformly to the invertible principal operator $A_{y,\eta}$. Writing $u = re^{i\theta}$, the allowed angles form two compact intervals and multiplication by $e^{i\theta}$ does not change singular values. Thus the least singular value of $P(u)/(1 + |u|)$ has a positive lower bound near the radial boundary at infinity. The remaining bounded portion of the closed sector is compact, and finite-parameter invertibility supplies the same conclusion there. This proves the resolvent estimate. □

Corollary 5.5 (fourteen-dimensional specialization)**.** *For $d = 13$ and $\eta = 1/4$, the threshold is*

$$Y = \frac{1}{13}.$$

Thus the pencil is sectorially parameter-elliptic for $0 < y^2 < 1/13$, has exactly linear critical resolvent growth at $y^2 = 1/13$, and has exactly one symmetric pair of real characteristic points for $y^2 > 1/13$.

6 Unnormalized products

Corollary 6.1 (two-parameter normalization)**.** *Let $x, y \in \mathbb{R}$ with $xy \neq 0$, and set*

$$m_{x,y}(A, B) = x[A, B] + iy(AB + BA).$$

and build the analogous pencil

$$P_{a,x,y,\eta}(u) = uD_{x,y} + aI + \tau - \eta u G_{x,y}\Gamma_{x,y}.$$

Define

$$C_{x,y}(d,\eta) = 4x^2 + 8\eta d\, x(x^2 - y^2) - 16\eta^2 d^2 x^2 y^2.$$

The trichotomy of Theorem 5.1 is controlled by the sign of $C_{x,y}$. In the negative case the shell is

$$u_*^2 = \frac{1 - a^2}{-C_{x,y}(d,\eta)}.$$

The positive case also has the sectorial estimate of Theorem 5.4.

Proof. Write

$$m_{x,y} = 2x\, m_z, \qquad z = \frac{y}{2x}, \qquad U = 2xu, \qquad \eta' = 2\eta x.$$

The unnormalized pencil is the normalized pencil with parameters (z, η', U). Multiplying $(1 + \eta' d)(1 - 4\eta' d z^2)$ by the positive factor $4x^2$ gives $C_{x,y}$. Since $U^2 = 4x^2 u^2$, both the real axis and the sector $\mathrm{Re}(u^2) \geq 0$ are preserved. □

7 Scope and reproducibility

The theorem is finite-dimensional and frozen-coefficient. It does not cover arbitrary Clifford modules, indefinite real forms, or general zero-order endomorphisms. Its content is the exact threshold, shell multiplicity, and local resolvent singularity for the explicit scalar/reversion pencil above. The use of Clifford monomial matrices is complementary to basis-free approaches to determinants, inverses, and Sylvester equations in Clifford algebras; see [5, 6].

The accompanying scripts implement Euclidean Clifford multiplication with bitmasks. The main verifier checks the principal spectrum, the even and odd return blocks, the determinant identities, the critical degree loss, and sampled singular values on both sides of the threshold. A separate direct-matrix audit constructs the full pencil without calling the block formulas and checks exact small-dimensional determinants, principal nullities, characteristic-shell ranks, and shell-resolvent residues. The symbolic checks are exact over rational or algebraic extensions; numerical scans are supplementary and are not used in the proofs. The programs, dependency file, and tested-environment record are collected as Online Resource 1.

8 Relation to prior work and originality boundary

The analytic and algebraic mechanisms used here belong to established frameworks. Parameter ellipticity supplies the surrounding resolvent language [1]; generalized-gradient decompositions motivate the grade-sensitive setting [2, 3, 4]; and general matrix-polynomial theory supplies the local language of finite and infinite spectral structure [7]. In particular, the distinction between determinant multiplicity and pole order is part of the general local theory of holomorphic matrix functions; see, for example, the semisimple first-order case in [8]. The contribution here is not a new version of those general principles, but their exact realization for the explicit Clifford pencil above.

There is also a terminological collision with the literature on "matrix pencils and generalized Clifford algebras." Pappacena associates a universal algebra to a prescribed minimal polynomial

of a matrix pencil [9], and Chapman and Kuo study that construction for monic polynomials [10]. Here the Euclidean Clifford algebra is fixed in advance, and the object of study is an operator pencil on $V_\perp^* \otimes \mathrm{Cl}_{d+1,\mathbb{C}}$ built from commutator–anticommutator multiplication, reversion, and a rectangular return term. These are different problems despite the shared words. Likewise, "two-sided Clifford" is used for Fourier transforms with left and right kernel factors [11]; that usage is unrelated to the finite-dimensional pencil considered here.

A targeted literature search completed on 14 July 2026 used the exact title, the defining product, the threshold coefficient, distinctive shell and resolvent phrases, arXiv and publisher-indexed records, and backward and forward citation chains around the works just cited. No earlier source was located that treats this specific family or states the coefficient

$$(1 + \eta d)(1 - 4\eta dY)$$

as a necessary and sufficient threshold simultaneously for every transverse Clifford grade. The originality claim is therefore deliberately narrow: the complete grade-indexed Schur reduction; the proof that the scalar obstruction is sharp for the full family; the two critical principal-nullity and degree-loss formulas; and the parity-resolved shell kernels, determinant crossings, and simple-pole residues. No novelty is claimed for Clifford multiplication or reversion themselves, for Sylvester and Woodbury identities, or for general matrix-pencil spectral theory. Since a negative search cannot prove priority, this paragraph records the strongest supportable literature position rather than an absolute claim of first discovery.

Statements and Declarations

Author contributions. Aaron Nagy is the sole author and is responsible for conceptualization, formal analysis, investigation, software, verification, writing, and approval of the final manuscript.

Funding. The author received no external funding for this work.

Competing interests. The author declares no competing interests.

Data and code availability. No research data were generated. Online Resource 1 contains the Python scripts, dependency file, and tested-environment record required to reproduce the reported symbolic and numerical checks.

Generative-AI disclosure. Generative-AI systems were used during exploratory derivation, code assistance, drafting, literature-search assistance, and editorial revision. The author selected the final statements, reworked the proofs, executed the verification suite, checked the cited sources, and accepts full responsibility for the manuscript.

References

[1] Mikhail S. Agranovich and Marko I. Vishik. Elliptic problems with a parameter and parabolic problems of general type. *Russian Mathematical Surveys* 19(3) (1964), 53–157. https://doi.org/10.1070/RM1964v019n03ABEH001149.

[2] Thomas P. Branson. Stein–Weiss operators and ellipticity. *Journal of Functional Analysis* 151 (1997), 334–383. https://doi.org/10.1006/jfan.1997.3162.

[3] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin Geometry*. Princeton University Press, Princeton, 1989. https://doi.org/10.1515/9781400883912.

[4] Mihaela Pilca. A new proof of Branson's classification of elliptic generalized gradients. *Manuscripta Mathematica* 136 (2011), 369–392. https://doi.org/10.1007/s00229-011-0430-4.

[5] Dmitry S. Shirokov. On computing the determinant, other characteristic polynomial coefficients, and inverse in Clifford algebras of arbitrary dimension. *Computational and Applied Mathematics* 40 (2021), Article 173. https://doi.org/10.1007/s40314-021-01536-0.

[6] Dmitry S. Shirokov. Basis-free solution to Sylvester equation in Clifford algebra of arbitrary dimension. *Advances in Applied Clifford Algebras* 31 (2021), Article 70. https://doi.org/10.1007/s00006-021-01173-0.

[7] Israel Gohberg, Peter Lancaster, and Leiba Rodman. *Matrix Polynomials*. Society for Industrial and Applied Mathematics, Philadelphia, 2009; reprint of the 1982 edition. https://doi.org/10.1137/1.9780898719024.

[8] Johannes M. Schumacher. Keldysh's theorem revisited. *Linear Algebra and its Applications* 730 (2026), 358–386. https://doi.org/10.1016/j.laa.2025.10.020.

[9] Christopher J. Pappacena. Matrix pencils and a generalized Clifford algebra. *Linear Algebra and its Applications* 313 (2000), 1–20. https://doi.org/10.1016/S0024-3795(00)00025-2.

[10] Adam Chapman and Jung-Miao Kuo. On the generalized Clifford algebra of a monic polynomial. *Linear Algebra and its Applications* 471 (2015), 184–202. https://doi.org/10.1016/j.laa.2014.12.030.

[11] Eckhard Hitzer. Two-sided Clifford Fourier transform with two square roots of -1 in $\mathrm{Cl}(p,q)$. *Advances in Applied Clifford Algebras* 24 (2014), 313–332. https://doi.org/10.1007/s00006-014-0441-9.